Research Statement

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§ 1. Introduction and Background

My work is in the field of logic and set theory. This field is concerned with statements independent of the standard axioms of set theory; examining their implications and their *strength*. In the foundations of mathematics, there is a natural ordering on the strength of axioms, not in the sense of material implication but instead the coarser measure of consistency strength: φ is less than or equal to ψ in consistency strength, $\varphi \leq_{\text{Con}} \psi$, if and only if the consistency of ψ implies the consistency of φ, regardless of whether ψ implies or refutes φ directly. This idea has its roots in Gödel incompleteness that says, generally speaking, φ is strictly weaker than the consistency of φ: $\varphi <_{\text{Con}} \text{Con}(\varphi)$ generally. *Equiconsistency* is therefore a topic of interest, examining what statements are of equal consistency strength, even if seemingly unrelated.

1 • 1. Question

What does the hierarchy of consistency strength—the graph of $\leq_{\text{Con}}$—look like?

Answering this question is difficult, but major progress has been made through *large cardinal axioms*. The idea is, much like how numbers are used to compare sizes between disparate things, large cardinal axioms provide a natural way of comparing consistency strength because they more-or-less form a linear hierarchy: if $\varphi =_{\text{Con}} \mathsf{LC}_0$ and $\psi =_{\text{Con}} \mathsf{LC}_1$ for large cardinal axioms LC_0 and LC_1, then φ and ψ can be easily compared based on the roughly linear order of consistency strength on LC_0 and LC_1. So a major goal in set theory is finding equiconsistencies with large cardinals. This is difficult, however.

The technique of *forcing* gives one direction. Forcing is the means by which we can expand from one universe of sets to another similar to a ring or field extension. In practice, from a large cardinal axiom LC true in the universe V, we often "force" some statement φ to be true in an extension $\mathrm{V}[G]$ to get $\varphi \leq_{\text{Con}} \mathsf{LC}$. Forcing is extremely powerful and popular as a technique in set theory for these kinds of relative consistency results. But equiconsistencies are fairly rare with this technique. Indeed, forcing is usually (provably) insufficient for the reverse direction, $\mathsf{LC} \leq_{\text{Con}} \varphi$. As a result, the inner model program of study—used to construct smaller universes $\mathrm{M} \subseteq \mathrm{V}$ in very precise, technical ways—becomes necessary, giving sophisticated techniques to find large cardinal properties from certain statements. This area is incredibly rich and concrete, able to make statements about the structure of the real numbers and connect them to large cardinal axioms, but is arguably less well studied compared with forcing due to its technical nature.

My work currently lies at an intersection of these techniques in forcing and inner model theory, although primarily focusing on the more popular area of large cardinals and forcing.

§ 2. Thesis Work

My thesis, a preprint of which can be found on my website here, explores the topic of *indestructibility*, where certain large cardinal properties are preserved after forcing with certain posets. Certainly not everything is changed by forcing, like canonical inner models. But that is due to their concreteness and absoluteness, something which large cardinals largely lack.

2 • 1. Question

Can large cardinals be immune (i.e. indestructible) to certain kinds of posets?

The thesis considers *weak indestructibility* for degrees of strength, meaning κ's ρ-strength being preserved by $< \kappa$-strategically closed, $\leq \kappa$-distributive posets, and gets an equiconsistency result in the large cardinal hierarchy.

There is a balance between the amount of large and small degrees of strength that can be weakly indestructible: one cannot have both all large[i] and all small[ii] degrees of strength as weakly indestructible when there are just two strong cardinals.[iii] Research elsewhere in the literature, explored by authors like Arthur Apter [1–3] and Joel Hamkins[2,4,5], has progressed largely by ignoring the small degrees of strength and considering *large* degrees of strength, where there are many strong cardinals with their strengths as weakly indestructible. My thesis, however, explores the weak indestructibility of all "small" degrees of strength rather than the "large" degrees for strong cardinals. Using core model techniques and forcing, I show the following equiconsistency result.

2 • 2. Theorem

The following are equiconsistent:

1. There are $\alpha + 1$-many strong reflecting strong cardinals.
2. There are $\alpha + 1$-many strong cardinals, and all $\kappa + 2$-strong cardinals κ have weakly indestructible $\kappa + 2$-strength.
3. There are $\alpha + 1$-many strong cardinals, and all cardinals κ that are strong up to λ, the next measurable limit of measurables, have weakly indestructible λ-strength.
4. There are $\alpha + 1$-many strong cardinals, and all cardinals κ that are strong up to the next measurable limit of measurables λ have weakly indestructible λ-strength.

and we can go beyond this with larger λs. The exact bound is unclear but is below the next cardinal $\mu > \kappa$ that is $\mu + 2$-strong. Here $\alpha \in \mathrm{Ord}$, or replacing "$\alpha + 1$" with "proper class". Curiously, this is strictly stronger than a proper class of weakly indestructible strongs whenever $\alpha > 0$.

The concept of a strong reflecting strong cardinals approaches the idea of a Woodin cardinal, and by generalizing the forcing method used in Theorem 2 • 2, we can extend the result *to* a Woodin. This is partially interesting because below a Woodin cardinal, the consistency strength difference between weak indestructibility for all small and all large degrees of strength equalizes; the two are

[i] Here I mean "large" in the sense of whenever κ is strong, κ's degrees of strength are (weakly) indestructible. Clearly if such a cardinal's large degrees of strength are (weakly) indestructible, then the small degrees are too. So the issue is whether the non-strong cardinals have their degrees of strength as (weakly) indestructible, and to what extent.

[ii] Here κ's ρ-strength is small if $\rho \leq \kappa + 2$, but really we can consider ρ below the next measurable limit of measurables (and even beyond this, but strictly below the next λ that is $\lambda + 2$-strong).

[iii] Indeed, one cannot have weak indestructibility for all degrees of strength when there is a strong cardinal κ and a $\lambda > \kappa$ that is $\lambda + 2$-strong.

equiconsistent.

2 • 3. Theorem

The following are equiconsistent

1. There is a Woodin cardinal δ.
2. δ is Woodin and every $<\delta$-strong cardinal has weakly indestructible strength.
3. δ is Woodin and every $\kappa < \delta$ that is $\kappa+2$-strong has weakly indestructible $\kappa+2$-strength.

The main idea behind Theorem 2 • 2 is that we do an Ord length, (reverse) Easton support iteration where at each stage κ, we attempt to destroy as much strength as possible using $<\kappa$-strategically closed, $\leq\kappa$-distributive posets. (Immunity to such posets is what makes κ's strength weakly indestructible.) This "trial by fire" is such that what remains is *de facto* weakly indestructible by forcing with such posets. Later on, we might resurrect degrees of strength, but this is fine, since such possibly destructible degrees are large; the small degrees remain weakly indestructible.

The difficult part of the proof is figuring out what sort of embedding $j : \mathrm{V} \to \mathrm{M}$ in the ground model V can be lifted into a λ-strong embedding in the generic extension $\mathrm{V}[G]$ for arbitrarily large λ. This is done by ensuring we have enough agreement about the preparation between V and M. This agreement is ensured by requiring that $\mathrm{cp}(j) = \kappa$ itself is λ-strong but also understands where the strong cardinals are below λ; hence we use strong cardinals reflecting the set of strongs. As a result, whereas it's common to deal with a preparation up to $\mathrm{cp}(j) = \kappa$ and get agreement $j(\mathbb{P}_\kappa)_\kappa = \mathbb{P}_\kappa$ up to the critical point, we need agreement up to $\lambda > \kappa$: $j(\mathbb{P}_\lambda)_\lambda = j(\mathbb{P}_\kappa)_\lambda = \mathbb{P}_\lambda$. Actually lifting this embedding requires a variety of techniques, e.g. passing to a kind of hull $\mathrm{N}[G] \preccurlyeq \mathrm{M}[G]$ and constructing a $\mathrm{N}[G]$-generic in $\mathrm{V}[G]$ that's actually generic over $\mathrm{M}[G]$.

The core model direction for the equiconsistency of Theorem 2 • 2, assuming weakly indestructible small degrees of strength, any cardinals κ that are $\kappa+2$-strong in V become strong in the core model K. And in fact, fully strong cardinals in V—with a strong cardinal above them—become strong reflecting strongs in K. Doing this, as with anything in inner model theory, requires a bit of technical checking, but the general idea is relatively straightforward and accessible modulo a few facts about the core model.

This same idea can be used with supercompacts, which fortunately an easier lift-up argument, but unfortunately, the inner model theory up to a supercompact is not understood at this time. And so the core model techniques necessary to show equiconsistencies do not exist yet. Nevertheless, this generates quite a lot of conjectures.

2 • 4. Conjecture

The equiconsistencies from Theorem 2 • 2 are also true for supercompacts in place of strongs.

§ 3. Proposed Research

One of the main motivations behind the thesis work is to explore indestructibility beyond the question of is it possible to get X-amount of indestructible strong/supercompact/strongly compact/etc. cardinals? Investigating interactions with other properties like reflection principles—as with my thesis regarding Woodin cardinals—can yield a rich bounty of new results in slightly different directions, including (but not limited to)

- To what extent can a cardinal's reflection properties in its embeddings be indestructible?
- Assuming there's some sort of answer to this, is there an upper bound to the amount of cardinals with such indestructible reflection?
- Is there a cardinal for which it's *not* possible to have certain universal weak indestructibility properties below it? In other words, how can we place limits on X above?
- What control do we have on what posets resurrect degrees of strength?
- To what extent do axioms about universal indestructibility affect combinatorial statements?
- Given answers to any of these questions, how can we use the techniques of inner model theory to prove equiconsistencies and place such statements in the large cardinal hierarchy?

These are just some directions projects around indestructibility can go. Further interaction with inner model theory, large cardinals, and forcing is to be expected. And as a result, this helps connect the sometimes disparate topics of set theory that are forcing and inner model theory.

Additionally, work has begun on exploring the Chang model of the form $\mathrm{L}(\mathrm{Ord}^\omega)$ by way of the variants $\mathrm{L}(\lambda^\omega)$ for certain $\lambda \in \mathrm{Ord}$ that resemble

$$\Theta = \sup\{\alpha \in \mathrm{Ord} : \exists f\ (f : \mathbb{R} \to \alpha \text{ is surjective})\}.$$

The issue primarily is that the Chang model is much less well explored than, say, $\mathrm{L}(\mathbb{R}) = \mathrm{L}(\omega^\omega) \subseteq \mathrm{L}(\mathrm{Ord}^\omega)$. There, we have standard ways of thinking approaching the model, axioms like determinacy and the regularity of Θ. There has been an extensive amount of research in about $\mathrm{L}(\mathbb{R})$, filling up books like the entire famous Cabal seminar series. Yet there are no such axioms for the Chang model despite the fact that its theory is unchanged by forcing assuming sufficient large cardinal assumptions.

3 • 1. Question

How should we approach the Chang model, i.e. what standard assumptions should we make?

So the work is primarily a part of a program to better understand how to approach the Chang model, and we currently are focusing on two simple assumptions about elements of a certain "thorn" sequence. The idea is to approach this thorn sequence like Θ in $\mathrm{L}(\mathbb{R})$, and to hopefully have some sort of analogy where Θ is to $\mathrm{L}(\mathbb{R})$ as the thorn sequence is to $\mathrm{L}(\mathrm{Ord}^\omega)$. The assumptions we focus on about the thorn sequence is a strong form of regularity, and a certain kind of bound on the complexity of powersets, both of which hold for Θ in the context of AD assuming Θ is regular.

To add a little more detail, adding a Cohen subset of Θ under the right conditions yields that $\Theta = \omega_2$ in the resulting model. Going beyond this is more difficult, but one may generalize the properties of Θ used in the context of determinacy to add more subsets and get, for example, $\sup\{\alpha : \exists f\ (f : \Theta^\omega \to \alpha \text{ is surjective})\}$ is forced to be ω_3, and so on. The idea is to define the sequence $þ_0 = \omega$,

$$þ_{n+1} = \sup\{\alpha : \exists f\ (f : þ_n^\omega \to \alpha \text{ is surjective})\},$$

and $þ_\alpha = \sup_{\xi<\alpha} þ_\xi$ for limit α. The hope is to see what reasonable assumptions[iv] ensure these $þ_\alpha$s can be forced to be the ω_αs, with perhaps a few skipped: $þ_0 = \aleph_0, þ_1 = \aleph_1^+ = \aleph_2, þ_2 = \aleph_2^+ = \aleph_3$, and so on in generic extensions of the Chang model or models of the form $\mathrm{L}(þ_\alpha^\omega)$ for some α. Work in preparation tells us we can get $þ_n = \aleph_{n+1}$ for $0 < n < \omega$, and calculate $þ_\alpha$ for $\alpha \leq \omega$ in this

[iv]E.g. a certain kind of strong sense of regularity similar to what's implied by GCH or what's true of Θ in the context of AD

way, but going beyond this to $\text{þ}_{\omega+1}$ is more difficult.

Under sufficient large cardinal assumptions, and under reasonable assumptions about the Chang model—in the form of strong regularity and how powersets are calculated—we conjecture that $\text{þ}_{\alpha+1} = \text{þ}_{\alpha}^{+}$ whenever $\alpha > 0$. This is true of $L(\mathbb{R})$, but the hope is that this could be extended to the Chang model.

All that said, I'm interested more broadly in projects related to large cardinals, forcing, and potential interactions of these with inner model theory.

Bibliography

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